Mathematical Foundations of Structured Space Theory (SST)

Supplement 3: Closure of the UPF Problem Set and Geometric Extensions from FCC Lattice Geometry

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April 2026
DOI: 10.5281/zenodo.19423573
Supplement to **Mathematical Foundations of Structured Space Theory (SST)** (DOI: 10.5281/zenodo.19080507)

Document Status and Supersession Note:

This document, like Supplements 1 and 2, accompanies the first version of *Mathematical Foundations of Structured Space Theory (SST)* (DOI: 10.5281/zenodo.19080507), together with *Structured Space Theory (SST), Version 3* (DOI: 10.5281/zenodo.18787608) and *Unified Pattern Framework (UPF)* (DOI: 10.5281/zenodo.17900825). The material published in Supplement 1 (DOI: 10.5281/zenodo.19141923), Supplement 2 (DOI: 10.5281/zenodo.19376061), and the present Supplement 3 (DOI: 10.5281/zenodo.19423573) **will be incorporated into future updated versions of these main documents; once those are issued, the supplements will be superseded**.

Abstract

This supplement derives six zero-parameter closures for the UPF Open Problems plus geometric extensions to the charged-lepton sector, quark sector, CKM mixing, gauge-coupling unification, Higgs mass, and proton stability, all within the present FCC/SpS worked realization. **First**, the proton–neutron mass splitting is shown to equal $\Delta m = m_n(R-1)/[z\cdot\ln(d^2_{NNN}/d^2_{NN})]$, predicting 1.2934 MeV versus 1.2933 MeV observed [Navas et al. 2024] (0.004%).

Second, the neutrino mass-squared splitting ratio is derived from a shared-node constraint mechanism, giving $\Delta m^2_{32}/\Delta m^2_{21} = z(z-1)/|F| = 33$, matching the observed 32.6 ± 0.9 [Navas et al. 2024] (0.5σ). Three neutrino flavors emerge from $z/|F| = 3$ octahedral axes. The absolute mass $m_3 = m_p(z+1)/(|F|z\cdot 12^9) = 49.25$ meV (0.5%), the PMNS mixing angles $\sin^2\theta_{12} = 4/13$ (0.2%), $\sin^2\theta_{23} = 7/13$ (1.4%), $\sin^2\theta_{13} = 1/44$ (3.3%), and the CP phase $\delta = -\pi/2$ (maximal, from the C_4 axis chirality) are all derived.

Third, the absolute proton mass is derived as $m_p = 24\cdot m_{Pl}/12^{19} \times R^2$, where the correction factor $R^2 = (K_h/K_n)^2$ arises from gravitational self-coupling. The prediction is 938.281 MeV versus 938.272 MeV observed [Navas et al. 2024] (0.001%). With this result, the isotope bridge equation becomes fully zero-parameter.

Fourth, the emission probability for a p6 pattern factorizes as $P = \delta_{\text{eff}}/24$, with $\delta_{\text{eff}} = 0$ for the ground state (proton stable) and $\delta_{\text{eff}} = 1/4$ for an excited state ($\tau \approx 10^{-21}$ s, nuclear γ timescale).

Fifth, the neutron drip line is $N_{\text{drip}} = Z + \lfloor (S - \text{extra}_p)/s \rfloor$, where S is the FCC surface void count, s = 2 for stable cores and s = 3 for unstable rank-0 cores. This reproduces the experimentally confirmed neutron drip lines tested here ($Z \le 10$; data from [Thoennessen 2016, Ahn et al. 2022]) within ±2 neutrons.

Sixth, the Weinberg angle is derived as $\sin^2\theta_W = (z/|F|)/(z+1) = 3/13 = 0.2308$, matching the observed 0.2312 [Navas et al. 2024] to 0.19%. The electromagnetic coupling normalization $\alpha^{-1}(0) = z^2 - (|F| + z/|F|) = 137$ (0.026%) and its running to $\alpha^{-1}(m_Z) = 128$ (0.08%) are derived from the two-step bond-response space and its excluded face-state and

screening sectors. All three gauge couplings unify at the Z mass with universal denominator $z^2 - |F|^2 = 128$: $\alpha_{EM} = 1/128$ (0.08%), $\alpha_s = (|F|^2-1)/128 = 15/128$ (0.1%), and the effective weak coupling $\alpha_W = \alpha_{EM} / \sin^2\theta_W = 13/384$ (0.2%), where the bare weak numerator $N_W = |F| = 4$ acquires the Weinberg mixing factor $(z+1)/(z/|F|) = 13/3$. The W-boson mass $m_W = m_p\, z^2/\sqrt{3} \times R^2 = 79.8$ GeV (0.7%), with $\sqrt{3}$ from the octahedral Laplacian eigenvalue $\lambda = 4$ multiplicity 3, and the Z-boson mass $m_Z = 91.0$ GeV (0.2%) follow directly.

Beyond the original UPF §20 problem set, the charged-lepton sector is also derived: the electron, muon, and tau masses follow from three stable delocalization topologies on the FCC lattice, with the universal factor $|F|^2 + 1 = 17$ (the vertex response algebra dimension) connecting all three generations. The proton-to-electron mass ratio $m_p/m_e = z^3(|F|^2 + 1)/|F|^2 = 1836.0$ matches observation to 0.008%, and the Koide ratio is satisfied to 0.01% with the geometric value $2/3 = 1 - |F|/z$ as the natural lattice interpretation. Fractional quark charges are derived from octahedral winding ($q_u = W/(z/|F|) = 2/3, q_d = -1/3$), all six quark masses follow from a charge-dependent coupling rule (0.6–1.3%), and the full CKM matrix is determined by four lattice parameters ($\lambda = 1/\sqrt{20}$, $A = 5/6$, $\delta = \arctan(5/2)$, $r = \sqrt{(3/20)}$), unifying with the PMNS matrix as external vs internal views of the same octahedral geometry. Proton stability is proved ($\delta_{eff} = 0$ by construction), and the Higgs mass $m_H = m_p(z^2 - z + 1) = 124.8$ GeV (0.24%) completes the electroweak sector.

Keywords — proton–neutron mass splitting; neutrino mass hierarchy; PMNS mixing angles; CP phase; absolute mass scale; charged lepton masses; Koide relation; quark masses; quark charges; CKM matrix; PMNS–CKM unification; proton stability; Higgs mass; strong coupling; unified gauge couplings; emission probability; neutron drip line; fine-structure constant; electromagnetic coupling; Weinberg angle; W-boson mass; electroweak hierarchy; FCC lattice; face-state overlap; bond-response kernel.

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1 Scope and Relation to Previous Work

This supplement extends the Mathematical Foundations of Structured Space Theory [Marson 2026a] and the Unified Pattern Framework [Marson 2026c] with six zero-parameter closures for the UPF Open Problems plus geometric extensions within the present FCC/SpS worked realization. Taken together with Mathematical Foundations — Supplement 1 [Marson 2026d] and Supplement 2 [Marson 2026e], the present supplement completes the closure of the Open Problems defined in UPF §20 at the present geometric level. In particular, this supplement addresses the remaining problems 9, 5, 8, 2, 3, and 7, while the principal gravitational open problems were closed in Supplement 2. Additionally, beyond closing the original UPF §20 problem set, this work derives the charged-lepton sector including the coupling-space partition that fixes the PMNS assignment, fractional quark charges and quark masses with a coherence-threshold lemma governing the generation hierarchy, the CKM matrix together with its geometric unification with PMNS mixing, gauge-coupling unification ($\alpha_s = 15/128$, effective $\alpha_W = 13/384$), a formal proton-stability theorem, and a lattice expression for the Higgs mass, thereby substantially strengthening SST and strongly confirming UPF as the microscopic closure engine of the theory.

All results use only quantities already established in the SST/UPF framework: the FCC coordination number $z = 12$, the face-state dimension $|F| = 4$ (Proposition 4.4.1), the gradient law E5, and the centroid circulation angular momentum ratio $R = 1.01145$ (UPF §18). No new axioms or adjustable local parameters are introduced, given the global SbS hierarchy address $k = 19$. The hierarchy depth $k = 19$ is taken as a structural property of our Sub-Structure within the Super-Structure, not as a quantity expected to emerge from local lattice stability. The derivational task is therefore not to obtain 19 from FCC micro-geometry, but to determine whether treating 19 as the SbS address yields a consistent and predictive cross-scale framework. Here “worked realization” means that SST is formulated generically, but the present supplement adopts a specific local FCC lattice realization together with a specific SpS hierarchical interpretation in order to obtain explicit quantitative results.

Some results presented here close previously open problems through derivational routes that differ from earlier developmental sketches in Mathematical Foundations, Supplement 1, Supplement 2, or related working notes. In such cases, the present supplement should be taken as the authoritative route for the closure claimed here, while the earlier documents remain valid where not explicitly superseded.

2 Schedule Enumeration

2.1 Valid p6 Schedules

A valid p6 cycling schedule is a 12-tick sequence visiting all 6 octahedral vertices exactly twice, with consecutive vertices always NN-connected (not antipodal), and cyclic. Exhaustive backtracking enumeration, fixing the first vertex to V0, yields 28,944 valid schedules, matching UPF §17.1.

2.2 Winding Number Distribution

The equatorial winding number W is computed using the signed-area (cross-product) method. The distribution is: $W = -8$: 420; $W = -6$: 720; $W = -4$: 4,728; $W = -2$: 4,080; $W = 0$: 9,048; $W = +2$: 4,080; $W = +4$: 4,728; $W = +6$: 720; $W = +8$: 420. Three-class structure: 9,948 / 9,048 / 9,948, confirming UPF §17.3.

2.3 Angular Momentum Quantization

The face-state centroid angular momentum $|L|$ takes exactly 9 discrete values. The squared magnitudes are integer multiples of 1/4: $|L|^2 \in \{0, 1/4, 2/4, 3/4, 4/4, 5/4, 6/4, 8/4, 9/4\}$. The quantum 1/4 is the face-state overlap fraction Δp from Proposition 4.4.1.

2.4 Centroid Circulation Ratio

Mean $|L|$ by winding class: $W = 0$: 0.8247; $W = \pm 2$: 0.8342. The ratio $R = 1.01145$ matches UPF §18.2 to 0.0001%.

3 Proton–Neutron Mass Splitting (Open Problem 9)

3.1 The Mass Splitting Formula

The neutron is heavier than the proton by $\Delta m = 1.2933$ MeV. In UPF, the neutron is a p6 pattern with $W = 0$ and the proton has $W = \pm 2$. The mass splitting formula is:

$$\Delta m = m_n \cdot (R - 1) / [z \cdot \ln(d^2_{nnn}/d^2_{nn})] \tag{1}$$

3.2 Derivation of the Denominator

The denominator $z \cdot \ln(2)$ arises from the SST gradient law E5: $\partial_r \ln(d) = -(1/c^2)\, \partial_r \Phi$. The potential difference between the NN shell (distance d) and the NNN shell (distance $d\sqrt{2}$) gives $\ln(d^2_{nnn}/d^2_{nn}) = \ln(2)$. The factor $z = 12$ distributes the asymmetry across all nearest-neighbor bonds. The logarithm appears because SST's gradient laws are intrinsically logarithmic — multiplicative transport, not additive (Supplement 1 §2.2).

3.3 Numerical Verification

$$\Delta m = 939.565 \times 0.01145 / 8.3178 = 1.2934 \ MeV$$

Observed: $\Delta m = 1.2933$ MeV. Match: **0.004%**. Zero free parameters.

Result 3.1. *$\Delta m = m_n(R-1)/[z \cdot \ln(d^2_{nnn}/d^2_{nn})] = 1.2934$ MeV. R from centroid circulation, z = 12 structural, ln(2) from E5 on the FCC shell ratio. Zero free parameters.*

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4 Neutrino Mass-Squared Splitting Ratio (Open Problem 5)

The derivation given here replaces earlier provisional closure sketches for this problem and is the route adopted in the present FCC/SpS worked realization.

4.1 Shared-Node Mass Mechanism

Neutrinos are modelled as two photon-like open patterns sharing a single lattice node. The mass arises from the face-state incompatibility at the junction: the shared node must simultaneously satisfy the propagation demands of both open patterns. From Proposition 4.4.1, the face-state overlap is $\Delta p = 1/4$.

4.2 Two-Level Coupling Structure

Level 1 (solar). The incompatibility is resolved within the $|F| = 4$ face-state directions at the shared node. At a given vertex, the face state has directions along 2 of the 3 octahedral axes. A constraint rotating between these two axes stays within the face-state structure. Coupling = 1.

Level 2 (atmospheric). The third octahedral axis has no face-state direction at the shared vertex. The constraint must propagate outward through $z - 1 = 11$ NN bonds, amplified by $z/|F| = 3$ at the next shell. Total: $(z-1) \times (z/|F|) = 33$.

4.3 The Mass-Squared Splitting Ratio

$$\Delta m^2_{32} / \Delta m^2_{21} = z(z-1)/|F| = 12 \times 11 / 4 = 33 \tag{2}$$

Result 4.1. *$\Delta m^2_{32}/\Delta m^2_{21} = z(z-1)/|F| = 33$. Observed: 32.6 ± 0.9 (0.5σ). Three flavors from $z/|F| = 3$ octahedral axes. Zero free parameters.*

4.4 Three Flavors from z/|F| = 3

The identity $z/|F| = 12/4 = 3$ equals the number of octahedral axes, which equals the number of independent vertex-pair couplings of the p6 pattern, which equals the number of neutrino flavors. Three flavors exist because the FCC lattice has 3 times more NN bonds than face-state directions per vertex.

4.5 Experimental Verification

Using PDG 2024 values: $\Delta m^2_{21} = (7.53 \pm 0.18) \times 10^{-5}$ eV^2; $|\Delta m^2_{32}| = (2.453 \pm 0.034) \times 10^{-3}$ eV^2. The observed ratio is 32.6 ± 0.9. The prediction 33 lies within 0.5σ.

4.6 Absolute Neutrino Mass Scale

The absolute mass of the heaviest eigenstate follows from the shared-node constraint energy. The neutrino is a shared-node open pair, and its mass is the self-energy of the trapped strain at the shared node — the face-state incompatibility. The self-energy propagator must traverse all z − z/|F| = 9 non-rotational channels (the 3 rotational channels carry flavor and maintain face-state coherence, leaving 9 for leakage). At each non-rotational step, the amplitude distributes over all z bond directions but only one continues the loop, giving a dilution factor of 1/z per channel. The total suppression is therefore $\left(\frac{1}{z}\right)^{z-\frac{z}{|F|}} = \left(\frac{1}{12}\right)^{9}$. The prefactor $\frac{z+1}{|F|z} = \frac{13}{48}$ arises because the neutrino couples through z+1 = 13 channels at the shared node, distributed over |F|z = 48 = $|O_h|$ symmetry-equivalent configurations (the full octahedral group O_h includes both rotational and inversion symmetries, and the shared-node constraint sees both because the face-state incompatibility does not distinguish a face from its inversion partner).

$$m^3 = \frac{mp(z+1)}{|F| \times z \times 12^{z-\frac{z}{|F|}}} = 49.25\ meV \tag{7}$$

Consistency check from the normal-hierarchy oscillation scale [Navas et al. 2024]: $m_3 \approx \sqrt{|\Delta m^2_{32}|} \approx 49.5$ meV. Match: 0.5%. The predicted $|\Delta m^2_{32}| = m_3^2 = 2.426 \times 10^{-3}$ eV^2 versus observed 2.453×10^{-3} eV^2 (1.1%). The sum $\Sigma m_\nu \approx 58$ meV satisfies the Planck cosmological bound $\Sigma m_\nu < 120$ meV.

4.7 PMNS Mixing Angles

The three PMNS mixing angles follow from vertex degree-of-freedom counting on the octahedron. The assignment of the three probabilities to the three geometric fractions is fixed by the charged-lepton delocalization hierarchy established in §6.2: each lepton generation accesses a specific layer of the coupling space, and the PMNS angles are the coupling fractions of these layers. The solar angle is the ratio of face-state directions to total coupling channels: $\sin^2\theta_{12}$ = |F|/(z+1) = 4/13. The atmospheric angle adds the rotation-axis degrees of freedom: $\sin^2\theta_{23}$ = (|F| + z/|F|)/(z+1) = 7/13, where 7 = |F| + z/|F| = 4 + 3 (face-state plus rotation axes). The reactor angle involves the chirality-odd coupling between the face-state and vertex sectors: $\sin^2\theta_{13}$ = 1/(|F|(z−1)) = 1/44. Note that the

Weinberg angle $\sin^2\theta_W = 3/13 = (z/|F|)/(z+1)$ shares the same denominator $z + 1 = 13$, and the three numerators (Weinberg 3, solar 4, atmospheric 7) sum to $14 = z + 2$ — the screening domain from Lemma 2.

Result 4.2. *m_3 = 49.25 meV (0.5%). $\sin^2\theta_{12} = 4/13 = 0.3077$ vs observed 0.307 ± 0.013 (0.2%). $\sin^2\theta_{23} = 7/13 = 0.5385$ vs observed 0.546 ± 0.021 (1.4%). $\sin^2\theta_{13} = 1/44 = 0.02273$ vs observed 0.0220 ± 0.0007 (3.3%). All observed values from [Navas et al. 2024]. Zero free parameters.*

4.8 CP Phase and Jarlskog Invariant

The leptonic CP phase δ_{CP} is determined by the chirality of the p6 winding on the octahedron. The p6 pattern with winding W = ±2 defines a handedness; adjacent face-state pairs are related by π/2 rotation around the C_4 (4-fold) symmetry axis. The CP phase is therefore $\delta_{CP} = -\pi/2$ (maximal CP violation), with the sign selected by the chirality of our universe. The Jarlskog invariant is $J = c_{12}\, c_{23}\, c^2_{13}\, s_{12}\, s_{23}\, s_{13} \sin\delta_{CP} = \sqrt{(9/13)} \times \sqrt{(6/13)} \times (43/44) \times \sqrt{(4/13)} \times \sqrt{(7/13)} \times \sqrt{(1/44)} \times (-1) = -0.0339$, consistent with the current experimental range [Navas et al. 2024]. The magnitude $|\delta_{CP}| = \pi/2$ is a geometric consequence of the 4-fold axis; only its sign is a spontaneous symmetry-breaking choice, analogous to the matter-antimatter asymmetry.

5 Absolute Mass Scale (Open Problem 8)

5.1 The Hierarchy Formula

UPF §16 observes that $m_p \approx 24 \cdot m_{Pl}/z^{19}$, matching the observed proton mass to 2.3%. The factor 24 is the number of direction-ticks per p6 cycle, z = 12 is the FCC coordination number, and k = 19 is the hierarchy depth — interpreted here as the cosmological address of our Sub-Structure within the Super-Structure cone geometry.

5.2 The Gravitational Self-Coupling Correction

The 2.3% residual is closed by a single correction factor. The proton is a W = ±2 pattern whose gravitational projection exceeds the neutral (W = 0) pattern by the factor $R = K_h/K_n = 1.01145$, derived from centroid circulation (UPF §18). Gravity couples to mass squared: the gravitational self-energy of a charged pattern is proportional to (gravitational projection)^2. The correction factor is therefore R^2:

$$m_p = 24 \cdot m_{pl} \,/\, 12^{19} \times R^2 \tag{3}$$

5.3 Numerical Verification

$$R^2 = 1.01145^2 = 1.023031$$

$$m_p = 917.16 \times 1.023031 = 938.281 \text{ MeV}$$

Observed: m_p = 938.272 MeV. Match: **0.001%.** Zero new local parameters, given the global SbS address k = 19.

Result 5.1. *$m_p = 24 \cdot m_{Pl}/12^{19} \times R^2 = 938.281$ MeV (0.001%). The correction R^2 is the gravitational self-coupling of the charged p6 pattern, with R derived from centroid circulation. Zero new local parameters, given the global SbS address k = 19.*

5.4 Consequences for the Bridge Equation

With $K_n = 24 \cdot G \cdot m_{Pl}/(gF^2 \cdot d^3) \times 12^{-19} \times R^2$, the isotope bridge equation (UPF §19) becomes fully zero-parameter. The original four empirical constants (K_h, K_n, α, β) are now all derived: R from centroid circulation, $\alpha = K_n/243$ and $\beta = K_n/378$ from overlap topology (UPF §19), and K_n itself from the hierarchy formula with R^2 correction.

6 Charged Lepton Masses

6.1 Generation Topology and Delocalization Rule

The three charged lepton generations correspond to the three stable delocalization topologies on the FCC lattice. A charged pattern can extend over n = 3, 2, or 0 spatial dimensions defined by the z/|F| = 3 octahedral axes. The case n = 1 (delocalization along a single axis) is excluded because it yields a propagating open mode (photon-like) rather than a massive localized pattern. The mass of each generation is suppressed by z^n relative to the proton mass scale, because a pattern delocalized over z^n lattice cells distributes its closure energy over that volume.

The universal factor connecting all three masses is $|F|^2 + 1 = 17$, the dimension of the vertex response algebra End(F) ⊕ 1: the $|F|^2 = 16$ ordered face-state transition pairs (initial × final) plus the scalar identity channel. This factor governs how the closure energy at each vertex distributes across the face-state interaction space.

6.2 Coupling-Space Partition and PMNS Assignment

Each delocalization dimension accesses a specific layer of the z+1 = 13 coupling channels at each vertex. The electron (n = 3, fully 3D delocalized) couples through the |F| = 4 face-state modes — the 3D lattice structure projected onto the four octahedral faces. The muon (n = 2, 2D delocalized) additionally accesses the z/|F| = 3 rotation-axis modes, each defined by a 2D rotation plane, giving |F| + z/|F| = 7 total modes. The tau (n = 0, localized) accesses all z+1 = 13 channels at its vertex. The three layers — face-state (4), rotation-axis (3), and vertex (6 = V) — partition the coupling space exactly:

$$|F| + \frac{z}{|F|} + V = 4 + 3 + 6 = 13 = z + 1$$

This identity is a structural theorem of the octahedral geometry, not a numerical coincidence. The neutrino flavor states are defined by the charged leptons (ν_e is the neutrino that couples to the electron), so the PMNS mixing angles are the coupling fractions of these layers: $\sin^2\theta_{12} = \frac{|F|}{z+1} = \frac{4}{13}$ the face-state fraction (electron sector), $\sin^2\theta_{23} =$

$\frac{|F|+\frac{z}{|F|}}{z+1} = \frac{7}{13}$ the non-vertex fraction (electron + muon sectors), and $\sin^2\theta_{13} = \frac{1}{|F|(z-1)} = \frac{1}{44}$ the chirality-odd fraction of the |F|×(z−1) = 44 A↔P couplings, where the winding W = 2 selects exactly one chirality-odd channel. The assignment of angles to geometric fractions is therefore fixed by the charged-lepton delocalization hierarchy, not assumed. This partition also explains why there are exactly three lepton generations: three layers in the coupling space, one per stable delocalization topology.

6.3 Electron Mass

The electron is a maximally delocalized charged pattern, extending over all 3 octahedral axes (n = 3). Its mass is suppressed by z^3 = 1728 relative to the proton, corrected by the vertex response factor $(|F|^2 + 1)/|F|^2$ = 17/16:

$$me = mp\frac{|F|^2}{z^3(|F|^2+1)};\ \frac{mp}{me} = \frac{z^3(|F|^2+1)}{|F|^2} = 1836.0 \tag{11}$$

Observed [Navas et al. 2024]: m_p/m_e = 1836.153. Match: 0.008%.

6.4 Muon Mass

The muon is delocalized over 2 axes (n = 2), giving z^2 suppression. The centroid circulation correction R = 1.01145 (UPF §18) enters because the 2D-delocalized pattern couples to the asymmetry between the two winding classes W = 0 and W = ±2, which is precisely what R measures:

$$m\mu = mp\,R\frac{|F|^2}{z^2} = mp\frac{R}{9} = 105.4\,MeV \tag{12}$$

Observed [Navas et al. 2024]: m_μ = 105.658 MeV. Match: 0.20%. The muon-to-electron ratio m_μ/m_e = R × z × $(|F|^2 + 1)$ = R × 204 = 206.3 (observed 206.77, 0.21%).

6.5 Tau Mass

The tau is a localized closed pattern with no spatial delocalization (n = 0). Its mass is comparable to the proton, with the ratio set by the vertex response algebra dimension divided by the non-rotational channel count:

$$m\tau = \frac{mp(|F|^2+1)}{z - \frac{z}{|F|}} = mp \times \frac{17}{9} = 1772\ MeV \tag{13}$$

Observed [Navas et al. 2024]: m_τ = 1776.86 MeV. Match: 0.26%.

6.6 Koide Relation and Geometric Interpretation

The three SST mass formulas satisfy the Koide relation $(m_e + m_\mu + m_\tau)/(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2$ = 0.6666 to 0.01% of the exact value 2/3. The geometric interpretation is 2/3 = 1 − |F|/z = 1 − 1/3: the fraction of the lattice coupling space remaining after removing one rotation-axis degree of freedom. However, exact algebraic equality with 2/3 is not a consequence of the three mass formulas as written, because the R factor in the muon formula prevents exact cancellation. The correct claim is that the Koide ratio is satisfied to 0.01%, with 2/3 = 1 − |F|/z providing the natural lattice interpretation.

Result 6.1. *$m_p/m_e = z^3(|F|^2+1)/|F|^2$ = 1836.0 (0.008%). $m_\mu = m_p\, R/9$ = 105.4 MeV (0.20%). $m_\tau = m_p \times 17/9$ = 1772 MeV (0.26%). Koide ratio satisfied to 0.01% with geometric value 2/3 = 1 − |F|/z. Zero free parameters.*

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7 Quark Charges, Masses, and CKM Mixing

7.1 Fractional Quark Charges from Octahedral Winding

The proton p6 has total winding W = 2, distributed over $\frac{z}{|F|} = 3$ axis-pairs. Each up-type axis carries winding $qu = \frac{W}{\frac{z}{|F|}} = \frac{2}{3}$. The total proton charge constraint ($2q_u + q_d = 1$) gives $qd = -\frac{1}{3}$. The unit of fractional charge, $\frac{1}{3} = \frac{1}{\frac{z}{|F|}}$, is the inverse of the color number: fractional charges exist because quarks carry fractions of the total winding, and the denominator is the number of octahedral axes.

7.2 Quark Mass Hierarchy

The proton mass is the total closure energy of the p6. Each of the $\frac{z}{|F|} = 3$ axis-pairs carries constituent mass $\frac{mp}{3} = 312.8$ MeV (the non-relativistic quark model value). The current quark mass is the constituent mass stripped of the cooperative binding enhancement z^2 = 144, the same B⨂B response-space dimension that governs α_{EM}. Therefore $mu = \frac{mp}{3z^2} = \frac{mp}{432} = 2.17$ MeV (observed 2.16 ± 0.07 MeV, 0.6%). The binding fraction $1 - \frac{1}{z^2} = 99.3\%$ matches the SM lattice QCD result that approximately 99% of the proton mass is binding energy.

The down quark has the opposite face-state orientation on the same axis. Its mass ratio to the up quark is $\frac{md}{mu} = \left|\frac{qu}{qd}\right| \times \frac{z+1}{z} = 2 \times \frac{13}{12} = \frac{13}{6} = 2.167$ (observed 2.162, 0.2%). The first factor is the charge ratio: the up quark's larger charge gives it more cooperative binding, so more of its mass goes into dressing and less remains as bare mass. The second factor $\frac{z+1}{z} = \frac{13}{12}$ is the self-coupling correction. Therefore $md = mu \times \frac{13}{6} = 4.71$ MeV (observed 4.67 ± 0.07 MeV, 0.8%).

The generation enhancement is governed by a coherence threshold. At each vertex, the face-state defines |F| = 4 modes, each carrying probability quantum $\Delta p = \frac{1}{|F|} = \frac{1}{4}$. A quark with charge q perturbs each mode by $|q|^2$. If $|q|^2 \geq \Delta p$, the perturbation saturates a full face-state mode, and all internal DOF respond coherently (superradiant scaling, N^2). If $|q|^2 < \Delta p$, the perturbation is sub-threshold, and each DOF responds independently (incoherent scaling, N). For the up quark, $|qu|^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9} > ¼ = \Delta p$, so the coupling is coherent and the generation enhancement scales quadratically with the channel count. For the down quark, $|qd|^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9} < ¼ = \Delta p$, so the coupling is incoherent and the enhancement scales linearly. The threshold $\Delta p = \frac{1}{|F|}$ is the same face-state quantum that governs nuclear bonding, emission probability, and proton stability. For the gen-1→2 transition, the down-type enhancement is $\frac{ms}{md} = (z+1) + \left(|F| + \frac{z}{|F|}\right) = 13 + 7 = 20$, giving m_s = 94.1 MeV (observed 93.4 ± 0.8, 0.7%). The up-type enhancement is $\frac{mc}{mu} = z \times \left(|F| + \frac{z}{|F|}\right)^2 =$

$12 \times 49 = 588$, giving m_c = 1277 MeV (observed 1270 ± 20, 0.6%). For the gen-2→3 transition, the down-type enhancement is $\frac{mb}{ms} = \frac{z(z-1)}{|F|} + z = 33 + 12 = 45$, giving m_b = 4235 MeV (observed 4180, 1.3%). The up-type enhancement is $\frac{mt}{mc} = z(z-1) + |F| = 132 + 4 = 136$, giving m_t = 173.7 GeV (observed 172.7, 0.6%). Quark mass conventions: u, d, s are MS-bar at μ = 2 GeV; c, b are MS-bar at μ = m_q; t is the direct (pole) mass [Navas et al. 2024].

7.3 CKM Parameters

The Cabibbo angle follows from the Gatto-Sartori-Tonin relation applied to the SST quark masses: $\sin\theta C = \sqrt{\frac{md}{ms}} = \frac{1}{\sqrt{20}} = 0.2236$ (observed 0.2253, 0.8%). The Wolfenstein A parameter is $A = \frac{V-1}{V} = \frac{5}{6} = 0.833$ (observed 0.814, 2.4%), where V = 6 is the number of octahedral vertices: a quark changing generation via W exchange can reach any of the other V−1 = 5 vertex positions out of V = 6 total. This gives $|Vcb| = A\lambda^2 = \left(\frac{5}{6}\right)\left(\frac{1}{20}\right) = \frac{1}{24} = 0.0417$ (observed 0.0415, 0.4%) — the same 24 = V×|F| from the emission formula $P = \frac{\delta eff}{24}$. The CKM CP phase is $\delta CKM = \arctan\left(\frac{\left(\frac{z}{|F|}\right)(V-1)}{V}\right) = \arctan\left(\frac{5}{2}\right) = 68.2°$ (observed 68.5°, 0.5%), reduced from the maximal PMNS value $|\delta| = \frac{\pi}{2}$ because quarks see the internal octahedral geometry scaled by $\frac{V-1}{V}$. The unitarity triangle radius is $r = \sqrt{\rho^2 + \eta^2} = \sqrt{\frac{\frac{z}{|F|}}{20}} = \sqrt{\frac{3}{20}} = 0.387$ (observed 0.384, 1.0%).

7.4 CKM Matrix Elements

From the four lattice parameters ($\lambda = \frac{1}{\sqrt{20}}, A = \frac{5}{6}, \delta = \arctan\left(\frac{5}{2}\right), r = \sqrt{\frac{3}{20}}$), all nine CKM matrix elements follow via the Wolfenstein parameterization. The diagonal $elements$ $|Vud| = 1 - \frac{\lambda^2}{2} = \frac{39}{40} = 0.975$ (observed 0.974, 0.1%), $|Vcs| = \frac{39}{40}$ (observed 0.973, 0.2%), and $|V_{tb}| \approx 1$ (observed 0.999, 0.0%). The first off-diagonal: $|Vus| = \frac{1}{\sqrt{20}} =$

0.224 (observed 0.225, 0.6%). The second off-diagonal: $|Vcb| = \frac{1}{24}$ (observed 0.042, 0.4%). The third off-diagonal: $|Vub| = A\lambda^3 r = \frac{5\sqrt{3}}{2400} = 0.0036$ (observed 0.0037, 2.2%). All elements match to 2.2% or better.

7.5 PMNS–CKM Unification

The PMNS and CKM matrices arise from the same octahedral geometry viewed from different perspectives. Leptons are free patterns on the lattice that see the full octahedral face-state structure from outside, giving large mixing angles with denominator z+1 = 13. Quarks are confined axis-pair components of the p6 that see only the internal bond structure, giving small mixing angles with denominator √20 = √(13+7). The CP phase scales from $|\delta PMNS| = \frac{\pi}{2}$ (full external C_4 chirality) to $\delta CKM = \arctan\left(\frac{5}{2}\right)$ (internal, scaled by $\frac{V-1}{V}$). The unification is that both matrices use the same structural numbers (z, |F|, V), but leptons access the full coupling space while quarks access the internal confinement space.

Result 7.1. *Quark charges $q_u = +2/3$, $q_d = -1/3$ derived from winding. Six quark masses from z, |F|, V: $m_u = m_p/432$ (0.6%), m_d (0.8%), m_s (0.7%), m_c (0.6%), m_b (1.3%), m_t (0.6%). CKM fully determined by $\lambda = 1/\sqrt{20}$, $A = 5/6$, $\delta = \arctan(5/2)$, $r = \sqrt{(3/20)}$: all 9 elements match to 2.2% or better. PMNS and CKM unified as external vs internal views of the same octahedral geometry.*

8 Emission Probability and Proton Stability (Open Problem 2)

The emission of a photon from a p6 pattern factorizes into a channel-selection factor and a break hazard. A symmetric p6 has V×|F| = 6×4 = 24 direction-ticks per cycle, and by symmetry each carries equal weight: $\pi a = \frac{1}{24}$. The break hazard is the closure deficit at the emitting vertex: $\frac{\Gamma break}{gF} = \delta eff$, where δ_{eff} measures the fraction of face-state not supported by the closure engine. For the ground-state proton, $\delta_{\text{eff}} = 0$ (perfect closure, no emission).

For an excited p6 with one direction already removed, $\delta eff = \Delta p = \frac{1}{|F|} = \frac{1}{4}$ — the same face-state quantum governing nuclear bonding (Proposition 4.4.1).

$$Pemit(channel\ a, per\ tick) = \frac{\delta eff}{24} \tag{4}$$

For an excited state ($\delta eff = \frac{1}{4}$): the per-channel rate is $\frac{gF}{96}$, giving an emission $timescale\ \tau = \frac{96}{gF} \approx 10^{-21}$ s. Emission is not a new force law; it is the probability that a closure-stable fixed point locally fails to maintain one internal direction long enough for that direction to reconfigure into an open Mode A photon.

Result 8.1. *$P_{emit} = \delta_{eff}/24$ per tick per channel. Ground state: $\delta_{eff} = 0$ (proton stable). Excited state: $\delta_{eff} = 1/4$ ($\tau \approx 10^{-21}$ s). Zero free parameters.*

8.1 Proton Stability Theorem

The ground-state p6 has $\delta_{eff} = 0$ exactly, by construction. The 28,944 valid schedules enumerated in §2 are defined as those for which every vertex transition is face-state-compatible. A nonzero δ_{eff} would require a face-state mismatch — an arriving bond demanding a face-state that the departing bond cannot satisfy — but such mismatches are excluded by the schedule validity condition. The net face-state winding over the complete 12-tick cycle is zero because the cycle returns to its initial vertex with its initial face-state (closed in both position space and face-state space). Therefore $Pemit = \frac{\delta eff}{24} = 0$ exactly: the proton is stable against single-photon emission. Proton decay would require a topological transition — breaking and reconnecting the Hamiltonian cycle — which demands energy exceeding the pattern's total closure energy. This is why the proton lifetime exceeds 10^{34} years.

9 Neutron Drip Lines (Open Problem 3)

The derivation given here replaces earlier provisional closure sketches for this problem and is the route adopted in the present FCC/SpS worked realization.

The neutron drip line follows from three lemmas. Lemma 1 (Bridge bond): an extra neutron beyond the α-cluster core bonds to the cluster through s shared octahedral vertices. For a core with cycle rank ≥ 1 (topologically stable) or with an extra proton providing stabilization, s = 2 (minimum phase-locked bridge bond). For a rank-0 core with no extra proton (e.g. the unstable $^2\alpha$ dimer), s = 3 (strong bond required for structural integrity). Lemma 2 (Surface exclusivity): each surface void site hosts at most one extra nucleon. A second would reuse the same shared-node pair, committing 2Δp = 1/2 of the face-state at the junction — exceeding the single-event overlap budget (Proposition 4.4.1). Lemma 3 (Drip line): the maximum neutron excess equals the surface attachment count divided by the sharing number.

$$Ndrip \;=\; Z \;+\; \left\lfloor \frac{S - extrap}{s} \right\rfloor \tag{5}$$

where S is the number of unoccupied FCC void sites adjacent to the compact α-cluster, $extra_p$ = Z mod 2, and s = 2 for stable cores or s = 3 for unstable rank-0 cores. The surface count S is computed from the FCC greedy packing that maximizes internal connectivity, and scales as $S \;\propto\; A^{\frac{2}{3}}$ from standard surface-to-volume geometry.

Result 9.1. *Neutron drip line: $N_{drip} = Z + \lfloor(S - extra_p)/s\rfloor$. Against experimentally confirmed drip lines ($Z \leq 10$): 9/9 within ± 2 neutrons. Against all estimates ($Z \leq 20$): 18/19 within ± 2, single outlier within the spread of standard nuclear mass models. Zero free parameters. Closed within the present FCC surface-attachment realization.*

Experimentally confirmed cases (Z ≤ 10; observed drip-line data from [Thoennessen 2016, Ahn et al. 2022]): He (pred 8, obs 8, Δ=0), Li (8, 8, 0), Be (10, 10, 0), B (13, 14, −1), C (17, 16, +1), N (17, 16, +1), O (20, 18, +2), F (20, 22, −2), Ne (24, 24, 0). Score: 9/9 within ±2. RMS residual: 1.2 neutrons.

10 Weak Interaction Hierarchy (Open Problem 7)

10.1 Unified Gauge Couplings

At the Z mass, all three Standard Model gauge couplings share the same denominator $z^2 - |F|^2 = (z-|F|)(z+|F|) = 8 \times 16 = 128$, differing only in the numerator — the number of face-state channels each force accesses:

$$\alpha i(mZ) = \frac{Ni}{z^2 - |F|^2} = \frac{Ni}{128}$$

where $N_{EM} = 1$ (the identity channel: a single photon exchange involves no face-state transition), $N_W = |F| = 4$ (the face-state modes: the W boson engages all $|F|$ face-state directions at each vertex), and $N_s = |F|^2 - 1 = 15$ (the independent face-state transition generators: each gluon exchange involves one of the $|F|^2 - 1$ non-identity transitions in the face-state algebra).

The strong coupling $\alpha_s(m_Z) = 15/128 = 0.1172$ matches the PDG world average 0.1171 ± 0.0017 to 0.1%. The decomposition $|F|^2 - 1 = (z/|F|)^2 - 1 + (|F| + z/|F|) = 8 + 7$ splits the strong coupling into 8 color generators (the $SU(z/|F|) = SU(3)$ gluon sector) plus 7 broken generators (the same $|F| + z/|F| = 7$ that appears in $\alpha^{-1}(0) = 144 - 7 = 137$ and $\sin^2\theta_{23} = 7/13$). The force hierarchy $\alpha_s > \alpha_W > \alpha_{EM}$ follows from $15 > 4 > 1$: face-state transitions outnumber face-states, which outnumber the identity. The bare weak numerator $N_W = |F| = 4$ gives $\alpha_W = 4/128 = 1/32 = 0.03125$, which is the local vertex-level coupling before electroweak mixing. The measured effective weak coupling at the Z mass includes the Weinberg rotation: $\alpha_W^{eff} = \alpha_{EM}/\sin^2\theta_W = (1/128)/(3/13) = 13/384 = 0.03385$, matching the PDG-derived value ~ 0.0338 to 0.2%. The effective numerator is therefore $(z+1)/(z/|F|) = 13/3 = 4.33$, consistent with the independently derived $\sin^2\theta_W = 3/13$.

10.2 Electromagnetic Coupling from the Two-Step Bond-Response Space

Let B be the 12-dimensional nearest-neighbor bond space at a lattice vertex. The electromagnetic coupling is bilinear in the gauge field, so the quadratic response kernel acts on $B \otimes B$. The unscreened channel budget is $\dim(B \otimes B) = z^2 = 144$ (Lemma 8.1). Two sectors are excluded from the electromagnetic budget.

First, the local p6 face-state occupies a 4-dimensional subspace of B, excluding $|F| = 4$ directions from the electromagnetic response (Lemma 8.2). These directions are committed to weak-sector coherence (the circular mode that generates W/Z coupling).

Second, at $k \to 0$ the 12 bonds partition into 3 four-bond rotational planes (one per C_4 axis). Each plane contributes exactly one gauge-invariant screening mode: the 4-bond C_4 orbit decomposes as $\chi_0 \oplus \chi_1 \oplus \chi_2 \oplus \chi_3$ (the four irreps of C_4), and only the symmetric component χ_0 couples to a spatially uniform photon field, since the nontrivial characters have amplitudes summing to zero (Lemma 8.3). The screening sector has dimension $z/|F| = 3$.

The two excluded sectors are geometrically independent: face-state bonds (body-diagonal directions at the octahedral vertex) always span at least 2 rotation planes, and therefore never coincide with any single coordinate-axis plane. Of all C(12,4) = 495 possible 4-bond subsets of B, only 3 (the three rotation-plane quartets) give a dependent system; physical face-state configurations are never among them (Lemma 8.4). The excluded subspace is therefore a direct sum, and the total excluded dimension is $|F| + z/|F| = 7$.

$$\alpha^{-1}(0) = z^2 - \left(|F| + \frac{z}{|F|}\right) = 144 - 7 = 137 \tag{9}$$

Observed [Navas et al. 2024]: $\alpha^{-1}(0) = 137.036$. Match: 0.026%. At high energy ($E \approx m_Z$), the C_4 degeneracy within each plane breaks and all $z = 12$ bond orientations contribute independently as screening channels:

$$\alpha^{-1}(mZ) = z^2 - (|F| + z) = 144 - 16 = 128 \tag{10}$$

Observed: $\alpha^{-1}(m_Z) = 127.9 \pm 0.02$ [Navas et al. 2024]. Match: 0.08%. The running $\Delta(\alpha^{-1}) = z - z/|F| = 9$ (observed: 9.1, 1.1% match) equals the same exponent that appears in the neutrino mass formula $m_3 \propto 1/12^9$: both count the non-rotational lattice channels that transition from collective to resolved screening. Note that $|F| + z = |F|^2 = 16$ on the FCC lattice (since $z/|F| = |F| - 1$), and the low-energy subtraction $7 = |F| + z/|F|$ is the atmospheric neutrino numerator from $\sin^2\theta_{23} = 7/13$.

Result 10.1. *$\alpha^{-1}(0) = z^2 - (|F| + z/|F|) = 137$ (0.026%). $\alpha^{-1}(m_Z) = z^2 - (|F| + z) = 128$ (0.08%). Running $\Delta(\alpha^{-1}) = z - z/|F| = 9$ (1.1%). Zero free parameters.*

10.3 Vertex Coupling Ratio

The electroweak sector decomposes into a local vertex-coupling ratio and a global mediator-mass hierarchy. The vertex coupling follows from the distinction between linear and circular emission modes on the octahedron: linear emission (photon) can launch from any of $|F| = 4$ face-state windows at each vertex, giving 24 total channels; circular emission (W/Z) requires a coherent matched pair, admitting only 1 window per vertex and 6 total channels. The bare vertex-level coupling ratio is therefore $\alpha_W/\alpha_{EM} = |F| = 4$. After electroweak mixing (Weinberg rotation), the effective weak coupling becomes $\alpha_W = \alpha_{EM}/\sin^2\theta_W = (1/128)/(3/13) = 13/384$, with effective numerator $(z+1)/(z/|F|) = 13/3 = 4.33$. Observed at the Z mass: $\alpha_W/\alpha_{EM} \approx 4.32$ (0.2% match for the effective coupling).

The Weinberg angle follows from counting degrees of freedom at a lattice vertex: $\sin^2\theta_W = (z/|F|)/(z+1) = 3/13 = 0.2308$, where $z/|F| = 3$ counts the octahedral rotation axes (independent planes of the circular mode) and $z + 1 = 13$ counts the total coupling channels (12 nearest-neighbor bonds plus 1 self-coupling).

$$sin^2\theta_W = \frac{\frac{z}{|F|}}{z+1} = \frac{3}{13} = 0.23077$$

Observed: 0.23122 ± 0.00003 (MS-bar scheme at the Z mass) [Navas et al. 2024]. Match: 0.19%. This immediately gives the W-to-Z mass ratio: $m_Z/m_W = 1/\cos\theta_W = \sqrt{(13/10)} = 1.1402$ versus observed 1.1345 (0.5% match). The local FCC face-state geometry thus fixes the electroweak coupling ratios and mixing angle.

10.4 W-Boson Mass Scale

The W boson is a circular-mode excitation requiring coherent engagement of all |F| face-state directions. The octahedral graph Laplacian L = D − A (degree matrix minus adjacency) has eigenvalue λ = 4 with multiplicity 3. These three degenerate eigenmodes are the three C_4 circular modes, one for each octahedral axis. The layer-17 inter-SbS energy $m_p \times z^2$ = 135.1 GeV is the total scale of this degenerate sector. The W boson corresponds to the amplitude of one resolved mode, so the mass is reduced by the square root of the degeneracy:

$$mW \;=\; mp\frac{z^2}{\sqrt{3}}\times\; R^2 \;=\; mp\; z^{\frac{3}{2}}|F|^{\frac{1}{2}}R^2 \;=\; 79.8\;GeV \tag{8}$$

The gravitational self-coupling correction R^2 applies as for the proton mass. Observed [Navas et al. 2024]: m_W = 80.4 GeV. Match: 0.7%. The Z-boson mass follows from the Weinberg angle: $m_Z = m_W/\cos\theta_W = m_W \times \sqrt{(13/10)}$ = 91.0 GeV versus observed 91.2 GeV (0.2%).

10.5 Low-Energy Interpretation

The effective low-energy weak coupling is suppressed through $G_F \propto \alpha_W/m_W^2$. With m_W now derived, this suppression is no longer an external input but a consequence of the FCC/SpS geometry. For the vertex-level comparison, $\alpha_W = g^2/(4\pi)$ with g the $SU(2)_L$ coupling, and $\alpha_{EM}(m_Z) = 1/127.9$, both from [Navas et al. 2024]; the ratio $\alpha_W/\alpha_{EM} \approx 4.34$ at the Z mass.

Result 10.2. *Unified gauge couplings $\alpha_i = N_i/128$ with $N_{EM} = 1$, $N_s = 15$: $\alpha_s = 15/128$ (0.1%). Bare $\alpha_W/\alpha_{EM} = |F| = 4$; effective $\alpha_W = 13/384$ (0.2%). $\sin^2\theta_W = 3/13$ (0.19%). $m_W = m_p\, z^2/\sqrt{3} \times R^2$ = 79.8 GeV (0.7%), $\sqrt{3}$ from octahedral Laplacian λ = 4 multiplicity 3. m_Z = 91.0 GeV (0.2%). $m_H = m_p(z^2-z+1)$ = 124.8 GeV (0.24%). Zero new local parameters, given k = 19.*

10.6 Higgs Mass

The Higgs boson is an excitation of the vacuum condensate — the ground state of the FCC lattice itself. Its mass is the energy cost of exciting the vacuum by one quantum. The vacuum energy involves the B⊗B bond-response space partitioned into three sectors: z^2 = 144 total channels (the full stiffness of the vacuum), minus z = 12 diagonal channels already committed to particle binding (the cooperative enhancement that gives quarks their z^2 mass dressing), plus 1 for the vacuum identity (the scalar ground state's zero-point mode). Therefore $m_H = m_p(z^2 - z + 1) = m_p \times 133 = 124.8$ GeV. Equivalently, $z^2 - z + 1 = (z^3+1)/(z+1) = 1729/13$: the lattice coordination volume distributed over the coupling channels.

Observed [Navas et al. 2024]: $m_H = 125.1$ GeV. Match: 0.24%. The formula carries no R^2 correction, because R^2 applies to propagating patterns (proton, W, Z), not to lattice states. The Higgs is a vacuum excitation — it does not propagate through the lattice, so it does not gravitationally self-couple. Verification: without R^2, match = 0.24%; with R^2, match worsens to 2.0%.

Result 10.3. *$m_H = m_p(z^2-z+1) = m_p \times 1729/13 = 124.8$ GeV (0.24%). No R^2 (vacuum condensate, not propagating pattern). Zero free parameters.*

11 Discussion

11.1 What This Closes

Problems 2, 3, 5, 7, and 9 are closed by local FCC geometric arguments within the present worked realization, while Problem 8 additionally uses the global SbS hierarchy address $k = 19$.

Open Problem 9 (proton–neutron mass splitting): closed. Zero free parameters, 0.004% match.

Open Problem 5 (neutrino sector): closed. Mass-squared ratio (33, 0.5σ), flavor count (3, exact), absolute mass scale (49.25 meV, 0.5%), PMNS mixing angles (0.2–3.3%), and CP phase ($\delta = -\pi/2$, consistent with T2K) derived within the present worked realization.

Open Problem 8 (absolute mass scale): effectively closed. The correction $C = R^2$ closes the 2.3% hierarchy gap to 0.001%. The bridge equation becomes fully zero-parameter. Zero new local parameters, given the global SbS address $k = 19$.

Charged lepton sector (beyond original UPF problem set): derived. Electron $m_p/m_e = z^3(|F|^2+1)/|F|^2 = 1836.0$ (0.008%), muon $m_\mu = m_p R/9$ (0.20%), tau $m_\tau = m_p \times 17/9$ (0.26%). Three stable delocalization topologies ($n = 3, 2, 0$) with universal factor $17 = |F|^2+1$. Koide ratio satisfied to 0.01% with geometric value $2/3 = 1 - |F|/z$. Zero free parameters.

Quark sector (beyond original UPF problem set): derived. Fractional charges $q_u = +2/3$, $q_d = -1/3$ from octahedral winding. Six quark masses from charge-dependent coupling rule: $m_u = m_p/432$ (0.6%), m_d (0.8%), m_s (0.7%), m_c (0.6%), m_b (1.3%), m_t (0.6%). CKM matrix fully determined by four lattice parameters: $\lambda = 1/\sqrt{20}$, $A = 5/6$, $\delta = \arctan(5/2)$, $r = \sqrt{(3/20)}$. All 9 CKM elements match to 2.2% or better. PMNS and CKM unified as external vs internal views of the same octahedral geometry. Zero free parameters. Quark mass conventions: MS-bar at $\mu = 2$ GeV for u, d, s; MS-bar at $\mu = m_q$ for c, b; direct (pole) mass for t.

Open Problem 2 (emission probability and proton stability): closed. $P_{emit} = \delta_{eff}/24$. Ground-state proton: $\delta_{eff} = 0$ by construction (proved), stable. Proton decay requires topological transition exceeding total closure energy. Excited state: $\delta_{eff} = 1/4$, $\tau \approx 10^{-21}$ s (nuclear γ timescale). Zero free parameters.

Open Problem 3 (drip-line predictions): closed within the present FCC surface-attachment realization. $N_{drip} = Z + \lfloor(S - extra_p)/s\rfloor$. Three lemmas: bridge bond sharing (s = 2 or 3), surface exclusivity (Prop 4.4.1), discrete surface count. 9/9 experimentally confirmed drip lines ($Z \leq 10$) within ±2. Zero new local parameters.

Open Problem 7 (electroweak sector): closed. Unified gauge couplings $\alpha_i = N_i/(z^2-|F|^2)$ with $N_{EM} = 1$, $N_W = |F| = 4$ (bare), $N_s = |F|^2-1 = 15$: $\alpha_{EM}(m_Z) = 1/128$ (0.08%), $\alpha_s(m_Z) = 15/128$ (0.1%). Effective weak coupling after Weinberg mixing: $\alpha_W = \alpha_{EM}/\sin^2\theta_W = 13/384$ (0.2%). Electromagnetic normalization $\alpha^{-1}(0) = 137$ (0.026%) and $\alpha^{-1}(m_Z) = 128$ (0.08%) from the two-step bond-response space. Weinberg angle $\sin^2\theta_W = 3/13$ (0.19%), $m_W = m_p z^2/\sqrt{3} \times R^2 = 79.8$ GeV (0.7%), $m_Z = 91.0$ GeV (0.2%), and $m_H = m_p(z^2-z+1) = 124.8$ GeV (0.24%) derived within the present worked realization. The Higgs mass carries no R^2 (vacuum condensate, not propagating pattern). Zero new local parameters, given the global SbS address k = 19.

11.2 Unified Coupling Structure

A single geometric quantity — the coupling factor $(z-1)/|F| = 11/4 = 2.75$ — appears in multiple derivations with different multipliers. The face-state dimension $|F| = 4$ governs nuclear bonding ($\Delta p = 1/4$), emission probability ($24 = V\times|F|$ channels), bare vertex coupling ($\alpha_W/\alpha_{EM} = |F| = 4$), effective weak coupling after Weinberg mixing ($\alpha_W^{eff} = 13/384$, effective numerator 13/3), surface exclusivity (drip line), face-state exclusion in the bond-response space (α^{-1} baseline), and the Weinberg angle denominator ($z+1 = 13$). The coordination $z = 12$ sets the shell ratio ln(2) in the mass splitting, the neutrino ratio ($\times(z-1) = 33$), the rotation-axis count ($z/|F| = 3$) in both the Weinberg angle and the low-energy screening sector, and the bond-resolved screening at high energy. The integers 3, 4, 7, 13 recur across sectors: $3/13 = \sin^2\theta_W$, $4/13 = \sin^2\theta_{12}$, $7/13 = \sin^2\theta_{23}$, $7 = |F| + z/|F|$ = low-energy excluded DOF in α^{-1} and broken generators in the strong coupling ($15 = 8 + 7$), $13 = z + 1$ = total coupling channels, and $3 + 4 + 7 = 14 = z + 2$ = the screening domain from Lemma 2.

11.3 Proof-Level Summary

The following table summarizes each result and its proof level:

Result	Section	Input / basis	Proof level
28,944 schedules verified	§2	Exhaustive backtracking	Computed
9,948/9,048/9,948 winding	§2.2	Cross-product method	Computed
$\lvert L\rvert^2$ quantized in units of 1/4	§2.3	Prop 4.4.1	Computed
R = 1.01145	§2.4	Centroid circulation (UPF §18)	Verified (0.0001%)
Δm = 1.2934 MeV	§3	R, z, ln(2) from E5	Derived (0.004%)
$\Delta m^2_{32}/\Delta m^2_{21} = 33$	§4	z(z-1)/|F|, two-level coupling	Derived (0.5σ)
Three neutrino flavors	§4.4	z/|F| = 3 octahedral axes	Geometric identity
m_p = 938.281 MeV	§5	$24\cdot m_{pl}/12^{19} \times R^2$	Derived (0.001%)
K_n fully derived	§5.4	Hierarchy + R^2 correction	Zero new local parameters
$P_{emit} = \delta_{eff}/24$	§8	Face-state counting + closure deficit	Derived
$N_{drip} = Z+\lfloor(S-extra_p)/s\rfloor$	§9	FCC surface counting + bridge lemmas	Derived in present realization
$\sin^2\theta_W = 3/13$	§10	Vertex DOF counting	Derived (0.19% match)
α_W = 13/384 (effective)	§10	$\alpha_{EM}/\sin^2\theta_W$ after Weinberg mixing	Derived (0.2% match)
$m_Z/m_W = \sqrt{(13/10)}$	§10	Corollary of θ_W	Derived (0.5% match)
m_3 = 49.25 meV	§4.6	Shared-node hierarchy suppression	Derived (0.5% match)
$\sin^2\theta_{12} = 4/13$	§4.7	|F|/(z+1) vertex DOF counting	Derived (0.2% match)
$\sin^2\theta_{23} = 7/13$	§4.7	(|F|+z/|F|)/(z+1) DOF counting	Derived (1.4% match)
$\sin^2\theta_{13} = 1/44$	§4.7	1/(|F|(z−1)) inter-bond mixing	Derived (3.3% match)
$\delta_{CP} = -\pi/2$	§4.8	C_4 axis chirality	Derived (consistent with T2K)
m_W = 79.8 GeV	§10.4	Laplacian λ=4 multiplicity 3 + layer-17 + R^2	Derived (0.7% match)
m_Z = 91.0 GeV	§10.4	Corollary of $m_W + \theta_W$	Derived (0.2% match)
$\alpha^{-1}(0) = 137$	§10.1	B⊗B budget − face-state − screening	Derived (0.026% match)
$\alpha^{-1}(m_Z) = 128$	§10.1	Bond-resolved screening corollary	Derived (0.08% match)
m_p/m_e = 1836.0	§6.3	$z^3(\lvert F\rvert^2+1)/\lvert F\rvert^2$ delocalization	Derived (0.008% match)
m_μ = 105.4 MeV	§6.4	$R\lvert F\rvert^2/z^2 = R/9$ delocalization	Derived (0.20% match)
m_τ = 1772 MeV	§6.5	$(\lvert F\rvert^2+1)/9$ localized pattern	Derived (0.26% match)
Koide ≈ 2/3	§6.6	1 − |F|/z = 1 − 1/3	Satisfied to 0.01%
$q_u=+2/3$, $q_d=-1/3$	§7.1	W/(z/|F|) winding distribution	Derived (exact)
m_u = 2.17 MeV	§7.2	$m_p/(3z^2)$ cooperative binding	Derived (0.6% match)
$m_d/m_u = 13/6$	§7.2	Charge ratio × coupling correction	Derived (0.2% match)
m_s, m_c, m_b, m_t	§7.2	Linear/quadratic charge-coupling	Derived (0.6–1.3%)
CKM (all 9 elements)	§7.4	λ=1/√20, A=5/6, δ=arctan (5/2), r=√(3/20)	Derived (≤2.2% match)
Proton stable (δ_{eff}=0)	§8.1	Face-state compatibility by construction	Proved (exact)
m_H = 124.8 GeV	§10.6	$m_p(z^2-z+1)$ vacuum condensate	Derived (0.24% match)
$\alpha_s(m_Z) = 15/128$	§10.1	$(\lvert F\rvert^2-1)/(z^2-\lvert F\rvert^2)$ unified formula	Derived (0.1% match)

Where the present supplement closes an open problem by a route different from that previously anticipated in developmental discussions, the closure should be understood as a refinement of the framework rather than a contradiction of the earlier supplements.

11.4 Scope and Outlook

The ten Open Problems defined in UPF §20 are closed at the present geometric level within the FCC/SpS worked realization. Beyond the original problem set, the charged-lepton sector, the quark sector (charges, masses, and CKM mixing), the Higgs mass, and proton stability have also been derived as extensions, with all results following from the same lattice invariants ($z = 12$, $|F| = 4$, $V = 6$, $R = 1.01145$). The PMNS and CKM mixing matrices are unified as external and internal views of the same octahedral geometry. Remaining work is no longer the closure of the original problem set, but the extension of these results to broader realizations, independent validation pathways, and eventual consolidation into updated SST, UPF, and Mathematical Foundations master documents.

The same small set of geometric invariants — $z = 12$, $|F| = 4$, $z/|F| = 3$, $z + 1 = 13$, $z - 1 = 11$, and $R = 1.01145$ — recurs across every sector: nuclear binding, mass splitting, neutrino oscillation, electroweak coupling, and boson masses. This recurrence, together with the absence of adjustable local parameters, strengthens the case that the Unified Pattern Framework is functioning as the microscopic closure engine of Structured Space Theory.

12 References

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